Arithmetic of hyperelliptic curves over local fields

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Setting

- $p \neq 2$
- $K/\mathbb{Q}_p$ finite
- C/K hyperelliptic of genus g
- $C : y^2 = c \cdot f(x) = c \prod_{r \in R}(x - r), \qquad R \subset \overline{K}, \quad c \in K^\times$
- $J = Jac(C)$

Clusters and cluster pictures

Cluster

A *cluster* of roots $\mathfrak{s}$ is a non-empty subset of R of the form

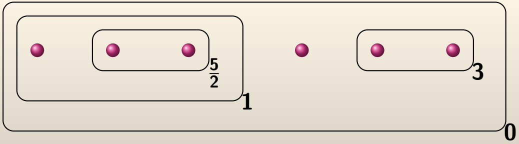
$$\mathfrak{s} = \{r \in R \mid v(r - z_{\mathfrak{s}}) \geq d\} = R \cap Disc(z_{\mathfrak{s}}, d),$$

with $z_{\mathfrak{s}} \in \overline{\mathbb{Q}_p}$, $d \in \mathbb{Q}$. The *depth* of $\mathfrak{s}$ is

$$d_{\mathfrak{s}} = \min_{r,r' \in \mathfrak{s}} v(r - r')$$

Example

$$C : y^2 = (x-p)(x^2-p^5)(x-2)(x-3)(x-3+p^3)$$



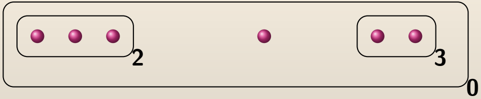
Local invariants of semistable C and J

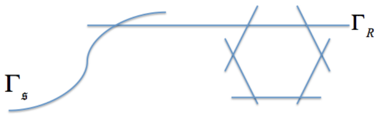
Results

- Necessary and sufficient conditions for C (and J) to be semistable;
- Minimal regular model C_{min} and its special fiber $\overline{C_{min}}$ + Frobenius action;
- Tamagawa group and Tamagawa number of J (A. Betts);
- Deficiency.

Example

$C : y^2 = (x-1)(x-1+p^2)(x-1-p^2)(x-2)x(x-p^3)$





$c_p = 6$ if $2 \in \mathbb{Q}_p^{\times 2}$, $c_p = 2$ otherwise.

Local invariants of C and J

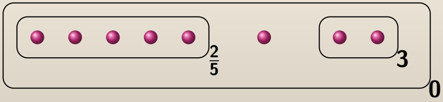
Results

- The ℓ-adic Galois representation $H^1_{et}(C/\overline{\mathbb{Q}}_p, \mathbb{Q}_\ell)$;
- Conductor;
- Root number;

Curve and Clusters

Let $p = 17$, $a = \sqrt{-p}$, and C be given by:

$$y^2 = (x^5 - p^2)(x-2)(x-1+p^3)(x-1-p^3)$$

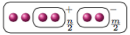
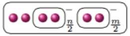
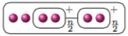
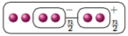
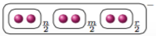


Frobenius

$$\begin{pmatrix} a & 0 & 0 & -a & & \\ 0 & 0 & a & -a & & \\ 0 & 0 & 0 & -a & & \\ 0 & a & 0 & -a & & \\ & & & & 1 & 0 \\ & & & & 0 & p \end{pmatrix}$$

Inertia

$$\begin{pmatrix} 0 & 0 & 0 & -1 & & \\ 1 & 0 & 0 & -1 & & \\ 0 & 1 & 0 & -1 & & \\ 0 & 0 & 1 & -1 & & \\ & & & & 1 & * \\ & & & & 0 & 1 \end{pmatrix}$$

Type	C	n_v	c_v	deficient	w_v
2		0	1	✗	1
1_n^+		1	n	✗	-1
1_n^-		1	n^*	✗	1
$I_{n,m}^{+,+}$		2	nm	✗	1
$I_{n,m}^{+,-}$		2	nm^*	✗	-1
$I_{n,m}^{-,-}$		2	n^*m^*	✗	1
$I_{n\text{-}n}^{+}$		2	n	✗	-1
$I_{n\text{-}n}^{-}$		2	n^*	✗	1
$U_{n,m,r}^{+}$		2	$nm+nr+mr$	✗	1
$U_{n,m,r}^{-}$		2	$(\frac{nm+nr+mr}{gcd(n,m,r)})^* \cdot gcd(n,m,r)^*$	$\begin{cases} ✓ & n,m,r \text{ odd} \\ ✗ & \text{else} \end{cases}$	1